BOUNDS FOR THE RANK OF THE FINITE PART OF OPERATOR K-THEORY AND POLYNOMIALLY FULL GROUPS

A Dissertation

by

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ABSTRACT

We derive a lower and an upper bound for the rank of the finite part of operator K-theory groups of maximal and reduced C^*-algebras of finitely generated groups. The lower bound is based on the amount of polynomially growing conjugacy classes of finite order elements in the group. The upper bound is based on the amount of torsion elements in the group. We use the lower bound to give lower bounds for the structure group $S(M)$ and the group of positive scalar curvature metrics $P(M)$ for an oriented manifold M.

We define a class of groups called "polynomially full groups" for which the upper bound and the lower bound we derive are the same. We show that the class of polynomially full groups contains all virtually nilpotent groups. As example, we give explicit formulas for the ranks of the finite parts of operator K-theory groups for the finitely generated abelian groups, the symmetric groups and the dihedral groups.

At the end, we discuss the possible directions to improve our results.

To my mother.

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NOMENCLATURE

G	Finitely generated group
G^{fin}	The set of torsion elements in the group G
G^{pol}	The set of elements in G with polynomially growing conjugacy class
p_g	The projection in $\mathbb{C}G$ obtained from a torsion element g in a group
$\mathbb{N}$	The set of positive integers
$\mathbb{R}_{>0}$	The set of positive real numbers
$\mathbb{R}_{\geq 0}$	The set of non-negative real numbers
$S(M)$	The structure group of a manifold M
$P(M)$	The group of positive scalar curvature metrics on a manifold M
$\ell^2 G$	Hilbert space of square summable complex valued functions on G
$\mathcal{B}(\mathcal{H})$	Bounded operators on the Hilbert space $\mathcal{H}$
$\mathbb{C}G$	Finitely supported complex valued functions on G
$C_r^* G$	Reduced group C^*-algebra
$C^* G$	Maximal group C^*-algebra
$C(h)$	Conjugacy class of the element h
$C_l(h)$	Elements in the conjugacy class of the element h with length l
$n_{h,l}$	Number of elements in the set $C_l(h)$
$C^G(h)$	Conjugacy class of the element h in the group G

$C_l^G(h)$	Elements in the conjugacy class of the element h in the finitely generated group G with length l

TABLE OF CONTENTS

1. INTRODUCTION

The K-groups are fundamental invariants for C^*-algebras. K-theory of C^*-algebras has important applications to geometry, topology, analysis and mathematical physics. If G is a countable group and C^*G is its maximal group C^*-algebra, then we have the Baum-Connes assembly homomorphism

$$\mu : K_i^G(EG) \to K_i(C^*G),\ i = 0, 1$$

where EG is the universal cover of the classifying space BG and $K_0^G(EG)$ is the equivariant K-homology group of EG. The strong Novikov conjecture states that the assembly map is injective. It is a consequence of the Baum-Connes conjecture [1]. The assembly map gives a way of constructing elements in the K_0 group of the maximal C^*-algebra C^*G, which is hard to construct directly. The strong Novikov conjecture gives an algorithm determining when higher index of an elliptic differential operator is non-vanishing. It implies the classical Novikov conjecture in topology, which states that the higher signatures are homotopy invariant.

In the case of G having torsion, detecting elements outside the image of the assembly map

$$\mu : K_0^G(EG) \to K_0(C^*G)$$

is important. Following the work of Weinberger and Yu [2] we can use such elements to measure the degree of topological non-rigidity for compact oriented manifolds within a given homotopy type and the size of all positive scalar curvature metrics on compact spin manifolds.

In our work, we detect those part of the K_0 group of the C^*-algebra C^*G, which cannot be detected by the assembly map

$$\mu : K_0^G(EG) \to K_0(C^*G).$$

The book by Connes [3] is an excellent resource for noncommutative geometry. In [4], Connes and Moscovici express the higher signatures in terms of the pairing between cyclic cohomology and K-theory and prove the Novikov conjecture for hyperbolic groups. Kasparov [5] introduces equivariant KK-theory and re-formulates the Novikov conjecture in the language of K-theory. Higson and Kasparov [6] prove the Baum-Connes conjecture for a-T-menable groups. Lafforgue [7] developes Banach KK-theory to prove Baum-Connes conjecture for strongly bolic groups with property RD. Mineyev and Yu [8] prove the Baum-Connes conjecture for hyperbolic groups and their subgroups. Yu [9] proves the coarse Baum-Connes conjecture for spaces which admit a uniform embedding into the Hilbert space.

This dissertation consists of six sections:

- **Section 1** is the introduction.
- In **section 2**, we give necessary background to state and prove our main result (Theorem 3.2.1).
- In **section 3**, we introduce numerical invariants which are the bounds for the ranks of the finite parts of the K_0 groups. We state and prove our main result at the end of the section.
- In **section 4**, we combine the results from Weinberger and Yu [2] and Theorem 3.2.1 to derive lower bounds for the ranks of the structure group and the group of positive scalar curvature metrics of manifolds.
- In **section 5**, we introduce the class of polynomially full groups. We show that finitely generated subgroups, products, finite extensions and images with finite kernels of polynomially full groups are also polynomially full. For a polynomially full group G, we show that

  $$K_0^{fin}(C_r^*G) \cong K_0^{fin}(C^*G) \cong \bigoplus_{i=1}^{\mathcal{F}_G} \mathbb{Z}.$$

  The class of polynomially full groups includes trivially all finite groups and finitely generated torsion-free groups. We show that it also includes all finitely generated virtually nilpotent groups. At the end of the section, we derive formulas for the number $\mathcal{F}_G$, where G is finitely generated abelian group, dihedral group, or symmetric group.

- **Section 6** is the conclusion. We give a summary of our work and discuss the possible ways to improve our results.

2. BACKGROUND

In this section, we introduce the notions necessary to state and prove our main result (Theorem 3.2.1).

2.1 C^*-algebras

In this subsection, we define the notion of C^*-algebras and give two important classes of C^*-algebras (reduced and maximal group C^*-algebras). A good introductory text on this subject is Murphy's book [10].

Definition 2.1.1. A C^*-algebra $\mathcal{A}$ is $\mathbb{C}$-algebra with a norm $\|.\|$ and an involution operation * satisfying the following properties:

- $\mathcal{A}$ is complete with respect to the norm $\|.\|$.
- For all $a, b \in \mathcal{A}$, we have $\|ab\| \leq \|a\| \, \|b\|$.
- For all $a \in \mathcal{A}$, we have $\|a^*a\| = \|a\|^2$.

Remark 2.1.2. Here we do not require that a $\mathbb{C}$-algebra contains the complex numbers. Hence a C^*-algebra may or may not contain a unit. The ones that contain 1 are called unital C^*-algebras.

Remark 2.1.3. It can easily be deduced that, for a C^*-algebra $\mathcal{A}$, we have $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$.

Example 2.1.4. For any compact topological space X, let $C(X)$ be the $\mathbb{C}$-algebra of continuous complex valued functions on X with operations of point-wise addition and multiplication. The complex algebra $C(X)$ is a C^*-algebra with the norm $\|f\| = \sup_{x \in X}\{\|f(x)\|\}$ and the *-operation being the point-wise conjugation operation.

Definition 2.1.5. A $\mathbb{C}$-algebra homomorphism between C^*-algebras $\phi : \mathcal{A} \to \mathcal{B}$ is called a *-homomorphism if it commutes with the *-operation (i.e. $\phi(a^*) = (\phi(a))^*$ for all $a \in \mathcal{A}$).

2.1.1 Reduced and Maximal Group C^*-algebras

In this subsubsection, we introduce two important classes of C^*-algebras called reduced and maximal group C^*-algebras.

For a finitely generated group G, let $\ell^2 G$ be the Hilbert space of square summable complex valued functions on G. The group G acts on $\ell^2 G$ by the following operation:

$$(g \cdot \xi)(h) = \xi(g^{-1}h)$$

for all $g, h \in G$ and $\xi \in \ell^2 G$. Extending this action $\mathbb{C}$-linearly, we get a faithful representation of $\mathbb{C}G$ as bounded operators on the Hilbert space $\ell^2 G$. In other words, we get a monomorphism (injective homomorphism) of $\mathbb{C}$-algebras $\mathbb{C}G \to \mathcal{B}(\ell^2 G)$.

Definition 2.1.6. Reduced group C^*-algebra $C_r^* G$ is defined to be the operator norm closure of the image of the above monomorphism.

For a finitely generated group G, we define a norm on $\mathbb{C}G$ as following:

$$\|.\|_{max} = \sup_\pi \{\|\pi(.)\|_{op}\},$$

where the supremum runs through all representations $\pi : \mathbb{C}G \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and $\|.\|_{op}$ is the operator norm.

Definition 2.1.7. The maximal group C^*-algebra C^*G is defined to be the closure of $\mathbb{C}G$ with respect to the norm $\|.\|_{max}$.

Remark 2.1.8. Since we use supremum over all representations of $\mathbb{C}G$ while we define the max-norm, every Cauchy sequence in $\mathbb{C}G$ with respect to the max-norm is a Cauchy sequence with respect to the norm we use to define the reduced group C^*-algebra $C_r^* G$. Hence, there is *-homomorphism $C^*G \to C_r^* G$ induced by the identity map on $\mathbb{C}G$.

2.2 K_0 Group of C^*-Algebras

In this subsection, we give the definition and basic properties of the K_0 group of C^*-algebras. A good introduction to the subject is the book of Rørdam, Larsen and Laustsen [11].

Given a unital C^*-algebra $\mathcal{A}$, let $M_n(\mathcal{A})$ be the algebra of square matrices of dimension n. The algebra $M_n(\mathcal{A})$ becomes a C^*-algebra with the *-operation defined as follows:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^* = \begin{pmatrix} a_{11}^* & a_{21}^* & \cdots & a_{n1}^* \\ a_{12}^* & a_{22}^* & \cdots & a_{n2}^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}^* & a_{2n}^* & \cdots & a_{nn}^* \end{pmatrix}.$$

To define a C^*-norm on $M_n(\mathcal{A})$, choose a Hilbert space $\mathcal{H}$ and an injective *-homomorphism $\phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Let $\phi_n\colon M_n(\mathcal{A}) \to \mathcal{B}(\mathcal{H}^n)$ be given by

$$\phi_n \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \phi(a_{11})\xi_1 + \cdots + \phi(a_{1n})\xi_n \\ \phi(a_{21})\xi_1 + \cdots + \phi(a_{2n})\xi_n \\ \vdots \\ \phi(a_{n1})\xi_1 + \cdots + \phi(a_{nn})\xi_n \end{pmatrix}, \quad \xi_j \in \mathcal{H}.$$

Define a norm on $M_n(\mathcal{A})$ by $\|a\| = \|\phi_n(a)\|$ for $a \in M_n(\mathcal{A})$. With these operations, $M_n(\mathcal{A})$ becomes a C^*-algebra, the norm is independent of the choice of representation ϕ provided that it is injective.

For a C^*-algebra $\mathcal{A}$, let $\mathcal{P}(\mathcal{A})$ be the set of projections in $\mathcal{A}$ (the elements $p \in \mathcal{A}$ which satisfy $p^2 = p^* = p$). Define $\mathcal{P}_n(\mathcal{A}) = \mathcal{P}(M_n(\mathcal{A}))$, and $\mathcal{P}_\infty(\mathcal{A}) = \bigcup\limits_{n=1}^{\infty} \mathcal{P}_n(\mathcal{A})$.

Define a relation $\sim_0$ on $\mathcal{P}_\infty(\mathcal{A})$ as follows. If $p \in \mathcal{P}_n(\mathcal{A})$ and $q \in \mathcal{P}_m(\mathcal{A})$, then we say $p \sim_0 q$ if there exists $v \in M_{m,n}(\mathcal{A})$ with $p = v^*v$ and $q = vv^*$. Where *-operation is defined as taking the transpose and applying * entry-wise.

Lemma 1. The relation $\sim_0$ on $\mathcal{P}_\infty(\mathcal{A})$ is an equivalence relation.

Proof. The only non-trivial part is the transitivity property. For $p \in \mathcal{P}_a(\mathcal{A}), q \in \mathcal{P}_b(\mathcal{A}), r \in \mathcal{P}_c(\mathcal{A})$

with $p \sim_0 q$ and $q \sim_0 r$ there exist $v \in M_{b,a}(\mathcal{A})$ and $w \in M_{c,b}(\mathcal{A})$ such that $p = v^*v$, $vv^* = q = w^*w$ and $r = ww^*$. Now we have $wv \in M_{c,a}(\mathcal{A})$ and $(wv)^*wv = v^*w^*wv = v^*qv = v^*vv^*v = p^2 = p$ and $wv(wv)^* = wvv^*w^* = wqw^* = ww^*ww^* = r^2 = r$. Hence, we get $p \sim_0 r$. $\square$

Define a binary operation $\oplus$ on $\mathcal{P}_\infty(\mathcal{A})$ by

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

One can show that the binary operation above respects the equivalence relation $\sim_0$. Hence it induces a binary operation on

$$\mathcal{D}(\mathcal{A}) \coloneqq \mathcal{P}_\infty(\mathcal{A}) / \sim_0 .$$

The set $\mathcal{D}(\mathcal{A})$ becomes an abelian semigroup with the operation $[p] + [q] = [p \oplus q]$ for $p, q \in \mathcal{P}_\infty(\mathcal{A})$, where $[p]$, $[q]$ and $[p \oplus q]$ are the equivalence classes represented by p, q and $p \oplus q \in \mathcal{P}_\infty(\mathcal{A})$.

Definition 2.2.1. For a unital C^*-algebra $\mathcal{A}$, we define $K_0(\mathcal{A})$ as the groupification of $(\mathcal{D}(\mathcal{A}), +)$ under the Grothendieck construction.

2.2.1 Finite Part of K_0 Groups

In this subsubsection, we introduce the notions of the finite part of the groups $K_0(C_r^*G)$ and $K_0(C^*G)$. In our main result, we give a lower and an upper bound for the ranks of the finite parts.

For a finitely generated group G let G^{fin} be the set of elements in G with finite order (set of torsion elements). For $g \in G^{\text{fin}}$ of order d, define $p_g = \frac{1+g+g^2+\cdots+g^{d-1}}{d} \in \mathbb{C}G$. It is easy to show that p_g is a projection in $\mathbb{C}G$. Since we have $\mathbb{C}G \subset C_r^*G$ and $\mathbb{C}G \subset C^*G$, p_g is a projection in C_r^*G and in C^*G. Thus it gives an element (with abuse of notation) $[p_g]$ in $K_0(C_r^*G)$ and in $K_0(C^*G)$.

Definition 2.2.2. The finite part of $K_0(C_r^*G)$ is the subgroup generated by the set $\{[p_g] \in K_0(C_r^*G) \colon g \in G^{\text{fin}}\}$. It is denoted by $K_0^{\text{fin}}(C_r^*G)$. The finite part $K_0^{\text{fin}}(C^*G)$ is defined similarly.

2.3 Polynomial Growth

In this subsection, we introduce the notion of polynomial growth in discrete metric spaces.

Let (X, d) be a discrete metric space. For $x \in X$ and $r \in \mathbb{N}$ let $B_r(x) := \{y \in X : d(x, y) \leq r\}$. We say (X, d) has polynomial growth (or (X, d) is polynomially growing) if there exists a polynomial $p \in \mathbb{R}[x]$ such that $|B_r(x)| \leq p(r)$ for all $r \in \mathbb{N}$. One can easily verify that this definition does not depend on the choice of the point $x \in X$. For a discrete metric space (X, d) and a subset $Y \subset X$, we say Y has polynomial growth if it has polynomial growth with respect to the metric d restricted to Y.

Example 2.3.1. Let G be a finitely generated group. Let $S \subset G$ be a finite generating set of G. The generating set S induces a norm on G as follows. For $g \in G$ we define $\|g\|_S := \min\{l \mid g = s_1 \cdot s_2 \cdots s_l,\ s_i \in S \cup S^{-1}\}$. This norm defines a metric as follows. For $g, h \in G$ define $d(g, h) := \|g^{-1}h\|_S$. Hence G becomes a discrete metric space. Thus, polynomially growing group or a conjugacy class having polynomial growth are defined accordingly. It can easily be shown that the notion of polynomial growth does not depend on the choice of the generating set S of G.

2.4 Dominating Functions, Seminorms, and Trace Functions*

In the first part of this subsection, we recall dominating functions from [13]. As Engel did in [14], using the dominating functions, we define a seminorm $\|.\|_{\mu,h}$ on $\mathbb{C}G$ for each $h \in G^{\mathrm{pol}}$, where G^{pol} is the set of elements with a polynomially growing conjugacy class. Using the seminorms and the operator norm, we complete $\mathbb{C}G$ and get a smooth dense subalgebra of C_r^*G. Recall that a subalgebra $\mathcal{A}$ of C_r^*G is called smooth if it is stable under holomorphic functional calculus and a subalgebra $\mathcal{A}$ of C_r^*G is stable under holomorphic functional calculus if for all $\alpha \in \mathcal{A}$ and f holomorphic in a neighborhood of the spectrum of α we have $f(\alpha) \in \mathcal{A}$.

We call that algebra $C_h^{pol}G$. In the second part of this subsection, we recall the trace functions on $\mathbb{C}G$ corresponding to an element in G. Using the properties of the seminorms, we lift τ_h to a trace function $\widetilde{\tau}_h$ on $C_h^{pol}G$ for each $h \in G^{\mathrm{pol}}$.

*Portions of this subsection are reprinted with permission from [12].

2.4.1 Dominating Functions and Seminorms

In this subsubsection, we recall the dominating functions, prove some properties about them, and using those functions, we define seminorms on $\mathbb{C}G$. Completing $\mathbb{C}G$ with respect to those seminorms and the operator norm, we construct smooth dense subalgebras $C_h^{pol}G$ of C_r^*G for $h \in G^{\mathrm{pol}}$.

In the following, we recall preliminary notions for the definition of the dominating functions. We use $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ for the sets of positive and non-negative real numbers, respectively.

Definition 2.4.1. Given $u \in \ell^2 G$ define $\mathrm{Supp}\ u := \{g \in G \ : \ u(g) \neq 0\}$. Now for all $S \subseteq G$ and $R \in \mathbb{R}_{\geq 0}$, define $B_R(S) := \{g \in G \ : \ d_w(g, S) \leq R\}$, where d_w is the metric induced by $\|.\|_w$. Define $\|u\|_S := (\sum_{g \in S} |u(g)|^2)^{\frac{1}{2}}$.

In the following, we recall the dominating function μ_A for an operator $A \in \mathcal{B}(\ell^2 G)$. We use these dominating functions to define some seminorms on $\mathbb{C}G$.

Definition 2.4.2. [14] For all $A \in \mathcal{B}(\ell^2 G)$ define $\mu_A : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ as

$$\mu_A(R) := \inf\{C \in \mathbb{R}_{>0} \ : \ \|Au\|_{G \setminus B_R(\mathrm{Supp}\ u)} \leq C \cdot \|u\|, \text{ for all } u \in \ell^2 G\}.$$

The following is a triangular inequality result we use at several places in our dissertation.

Lemma 2.4.3. *[12] For all $A, B \in \mathcal{B}(\ell^2 G)$ and $R \in \mathbb{R}_{>0}$ we have $\mu_{A+B}(R) \leq \mu_A(R) + \mu_B(R)$.*

Proof. For all $R \in \mathbb{R}_{>0}$ and $u \in \ell^2 G$, we have

$$\begin{aligned} \|(A+B)u\|_{G \setminus B_R(Supp\ u)} &\leq \|Au\|_{G \setminus B_R(Supp\ u)} + \|Bu\|_{G \setminus B_R(Supp\ u)} \\ &\leq \mu_A(R) \cdot \|u\| + \mu_B(R) \cdot \|u\| \\ &= (\mu_A(R) + \mu_B(R)) \cdot \|u\| \ . \end{aligned}$$

Thus, we get $\mu_{A+B}(R) \leq \mu_A(R) + \mu_B(R)$ for all $R \in \mathbb{R}_{>0}$. $\square$

In the following, we estimate the dominating function with the operator norm.

Lemma 2.4.4. *[12] For all $A \in \mathcal{B}(\ell^2 G)$, we have $\mu_A(R) \leq \|A\|_{op}$ for all $R \in \mathbb{R}_{>0}$.*

Proof. For all $A \in \mathcal{B}(\ell^2 G),\ R \in \mathbb{R}_{>0},\ u \in \ell^2 G$, we have

$$\|Au\|_{G \setminus B_R(Supp\ u)} \leq \|Au\| \leq \|A\|_{op} \cdot \|u\|.$$

Thus, we get $\mu_A(R) \leq \|A\|_{op}$. □

In the following, we use the previous estimate to show that convergence in the operator norm implies point-wise convergence in the dominating functions. We use this result in the proof of the smoothness of the subalgebras $C_h^{pol}G$ of C_r^*G for $h \in G^{\mathrm{pol}}$.

Lemma 2.4.5. *[12] Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of operators in $\mathcal{B}(\ell^2 G)$ converging (in $\|.\|_{op}$ norm) to $A \in \mathcal{B}(\ell^2 G)$. Then $\{\mu_{A_n}\}_{n=1}^{\infty}$ converges to μ_A point-wise.*

Proof. Given $R \in \mathbb{R}_{>0}$, we have

$$\begin{aligned}
\mu_{A_n}(R) &= \mu_{A+(A_n-A)}(R) \\
&\leq \mu_A(R) + \mu_{A_n-A}(R) \\
&\leq \mu_A(R) + \|A_n - A\|_{op}\,.
\end{aligned}$$

Similarly, we get $\mu_A(R) \leq \mu_{A_n}(R) + \|A - A_n\|_{op}$. Thus, we have

$$\mu_A(R) - \|A - A_n\|_{op} \leq \mu_{A_n}(R) \leq \mu_A(R) + \|A_n - A\|_{op}.$$

Hence, we get $\lim_{n\to\infty} \mu_{A_n}(R) = \mu_A(R),\ \forall R \in \mathbb{R}_{>0}$. □

Remark 2.4.6. Actually we have uniform convergence of $\{\mu_{A_n}\}_{n=1}^{\infty}$ to μ_A. However, point-wise convergence is enough for our purposes.

In the following, we define the seminorms we use to build the smooth dense subalgebras $C_h^{pol}G$ of C_r^*G for $h \in G^{\mathrm{pol}}$.

Definition 2.4.7. Recall that given $h \in G^{\mathrm{pol}}$ $\exists C_h \in \mathbb{R}_{>0}$ and $d_h \in \mathbb{N}$ such that $\forall l \in \mathbb{N}$ we have $n_{h,l} \leq C_h \cdot l^{d_h}$. Let b_h be a natural number greater than or equal to $\frac{d_h}{2} + 2$. Now define *

$$\|A\|_{\mu,h} := \inf\{D \in \mathbb{R}_{>0} \mid \mu_A(R) \leq D \cdot R^{-b_h} \ \forall R \in \mathbb{R}_{>0}\}.$$

Lemma 2.4.8. *[12] $\|.\|_{\mu,h}$ is a seminorm on $\mathbb{C}G$.*

Proof. For all $A \in \mathbb{C}G$, we have obviously $\|A\|_{\mu,h} \geq 0$.

Now for all $A, B \in \mathbb{C}G$ and $R > 0$ we have

$$\begin{aligned}\mu_{A+B}(R) &\leq \mu_A(R) + \mu_B(R)\\ &\leq \|A\|_{\mu,h}R^{-b_h} + \|B\|_{\mu,h}R^{-b_h}\\ &= (\|A\|_{\mu,h} + \|B\|_{\mu,h})R^{-b_h}.\end{aligned}$$

Therefore $\|A + B\|_{\mu,h} \leq \|A\|_{\mu,h} + \|B\|_{\mu,h}$. Hence, $\|.\|_{\mu,h}$ is a seminorm on $\mathbb{C}G$. □

In the following, we define our main gadget (a smooth dense subalgebra of C_r^*G). We use the properties of the seminorm to lift the trace function τ_h (originally on $\mathbb{C}G$) to this subalgebra of C_r^*G.

Definition 2.4.9. For each $h \in G^{\mathrm{pol}}$, we define $C_h^{pol}G$ as the completion of $\mathbb{C}G$ with respect to the norm $\|.\|_{op}$ and the seminorm $\|.\|_{\mu,h}$.

Since $C_h^{pol}G$ contains $\mathbb{C}G$, it is dense (in the operator norm) in C_r^*G.

In the following, we show that $C_h^{pol}G$ is an algebra over the complex numbers. The only nontrivial part is to show that it is closed under multiplication.

Lemma 2.4.10. *[12] $C_h^{pol}G$ is an algebra over $\mathbb{C}$.*

*We use a notation different than in [14].

Proof. Given $A, B \in C_h^{pol}G$, there exist sequences $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ in $\mathbb{C}G$ converging (in both norms) to A and B respectively.

The only nontrivial part is to show that $\lim_{n\to\infty}\|AB - A_nB_n\|_{\mu,h} = 0$. We have

$$\begin{aligned}\|AB - A_nB_n\|_{\mu,h} &= \|(AB - A_nB) + (A_nB - A_nB_n)\|_{\mu,h} \\ &= \|(A - A_n)B + A_n(B - B_n)\|_{\mu,h} \\ &\le \|(A - A_n)B\|_{\mu,h} + \|A_n(B - B_n)\|_{\mu,h}\ .\end{aligned}$$

We only show $\lim_{n\to\infty}\|(A - A_n)B\|_{\mu,h} = 0$:

Let $C_n = A - A_n$ then, for all $R \in \mathbb{R}_{>0}$, we have (the first inequality below is from [13, Prop. 5.2])

$$\begin{aligned}\mu_{C_nB}(R) &\le 2\|C_n\|_{op}\mu_B(R/2) + \|B\|_{op}\mu_{C_n}(R/2) + 2\mu_{C_n}(R/2)\mu_B(R/2) \\ &\le 2\|C_n\|_{op}\|B\|_{\mu,h}(R/2)^{-b_h} + \|B\|_{op}\|C_n\|_{\mu,h}(R/2)^{-b_h} + 2\|C_n\|_{\mu,h}\|B\|_{\mu,h}(R/2)^{-2b_h} \\ &= \{2^{b_h+1}\|C_n\|_{op}\|B\|_{\mu,h} + 2^{b_h}\|C_n\|_{\mu,h}\|B\|_{op} + 2^{2b_h+1}\|C_n\|_{\mu,h}\|B\|_{\mu,h}R^{-b_h}\}R^{-b_h}\ .\end{aligned}$$

Now if $R \ge 1$, then

$$\begin{aligned}&2^{b_h+1}\|C_n\|_{op}\|B\|_{\mu,h} + 2^{b_h}\|C_n\|_{\mu,h}\|B\|_{op} + 2^{2b_h+1}\|C_n\|_{\mu,h}\|B\|_{\mu,h}R^{-b_h} \le \\ &2^{b_h+1}\|C_n\|_{op}\|B\|_{\mu,h} + 2^{b_h}\|C_n\|_{\mu,h}\|B\|_{op} + 2^{2b_h+1}\|C_n\|_{\mu,h}\|B\|_{\mu,h}\ .\end{aligned}$$

Let $D_n = 2^{b_h+1}\|C_n\|_{op}\|B\|_{\mu,h} + 2^{b_h}\|C_n\|_{\mu,h}\|B\|_{op} + 2^{2b_h+1}\|C_n\|_{\mu,h}\|B\|_{\mu,h}$.

If $0 < R < 1$, then

$$\mu_{C_nB}(R) \le \|C_nB\|_{op} \le \|C_n\|_{op}\|B\|_{op} \le \|C_n\|_{op}\|B\|_{op}\cdot R^{-b_h}.$$

So for all $R \in \mathbb{R}_{>0}$, we have $\mu_{C_nB}(R) \le \max\{D_n, \|C_n\|_{op}\|B\|_{op}\}\cdot R^{-b_h}$. Hence, we get $\|C_nB\|_{\mu,h} \le$

$\max\{D_n, \|C_n\|_{op}\|B\|_{op}\}$. Since we have

$$\lim_{n\to\infty} D_n = \lim_{n\to\infty} \|C_n\|_{op}\|B\|_{op} = 0,$$

we get $\lim_{n\to\infty}\|C_nB\|_{\mu,h} = 0$.

Similarly, we can show that $\lim_{n\to\infty}\|A_n(B - B_n)\|_{\mu,h} = 0$. Thus, we get

$$\lim_{n\to\infty}\|AB - A_nB_n\|_{\mu,h} = 0.$$

Hence, we have $\lim_{n\to\infty} A_nB_n = AB$. So $AB \in C_h^{pol}G$. Thus, $C_h^{pol}G$ is an algebra. $\square$

In the following, we are giving an estimate that is used in the proof of smoothness of $C_h^{pol}G$. It can be proven by induction on n.

Lemma 2.4.11. *[14] Given $A \in C_h^{pol}G$ and $n \in \mathbb{N}$, we have*

$$\mu_{(\mathrm{Id}-A)^n}(R) \leq \sum_{k=1}^{n-1} 5^k \|\mathrm{Id}-A\|_{op}^{n-1} \mu_A\Big(\frac{R}{2^k}\Big).$$

In the following, we show that $C_h^{pol}G$ is a smooth subalgebra of C_r^*G.

Lemma 2.4.12. *[14] $C_h^{pol}G$ is closed under holomorphic functional calculus.*

Proof. Given $A \in C_h^{pol}G$ with $\|\mathrm{Id}-A\|_{op} < \epsilon$, where $\epsilon = \frac{1}{2}\frac{1}{5\cdot 2^{b_h}}$. We have

$$
\begin{aligned}
\mu_{A^{-1}-\sum_{n=0}^{N}(\mathrm{Id}-A)^n}(R) &\leq \sum_{n=N+1}^{\infty} \mu_{(\mathrm{Id}-A)^n}(R) \\
&\leq \sum_{n=N+1}^{\infty} \sum_{k=1}^{n-1} 5^k \epsilon^{n-1} \mu_A\Big(\frac{R}{2^k}\Big) \\
&= \sum_{k=1}^{N} \Big\{ 5^k \mu_A\Big(\frac{R}{2^k}\Big) \cdot \sum_{n=N+1}^{\infty} \epsilon^{n-1} \Big\} + \sum_{k=N+1}^{\infty} \Big\{ 5^k \mu_A\Big(\frac{R}{2^k}\Big) \cdot \sum_{n=k+1}^{\infty} \epsilon^{n-1} \Big\} \\
&= \frac{\epsilon^N}{1-\epsilon} \sum_{k=1}^{N} 5^k \mu_A\Big(\frac{R}{2^k}\Big) + \frac{1}{1-\epsilon} \sum_{k=N+1}^{\infty} (5\epsilon)^k \mu_A\Big(\frac{R}{2^k}\Big) \\
&\leq \frac{\epsilon^N \cdot \|A\|_{\mu,h}}{1-\epsilon} \cdot R^{-b_h} \cdot \sum_{k=1}^{N} (5 \cdot 2^{b_h})^k + \frac{\|A\|_{\mu,h}}{1-\epsilon} \cdot R^{-b_h} \cdot \sum_{k=N+1}^{\infty} (5 \cdot \epsilon \cdot 2^{b_h})^k \\
&= \frac{\|A\|_{\mu,h}}{1-\epsilon} \cdot \Big\{ \epsilon^N \sum_{k=1}^{N} (5 \cdot 2^{b_h})^k + \sum_{k=N+1}^{\infty} (\frac{1}{2})^k \Big\} \cdot R^{-b_h} .
\end{aligned}
$$

Thus, we have $\|A^{-1} - \sum_{n=0}^{N}(\mathrm{Id}-A)^n\|_{\mu,h} \leq \frac{\|A\|_{\mu,h}}{1-\epsilon} \cdot \{\epsilon^N \sum_{k=1}^{N}(5 \cdot 2^{b_h})^k + \sum_{k=N+1}^{\infty}(\frac{1}{2})^k\}$. Now since $\lim_{N\to\infty}\{\epsilon^N \sum_{k=1}^{N}(5 \cdot 2^{b_h})^k + \sum_{k=N+1}^{\infty}(\frac{1}{2})^k\} = 0$, we get

$$
\lim_{N\to\infty} \|A^{-1} - \sum_{n=0}^{N}(\mathrm{Id}-A)^n\|_{\mu,h} = 0.
$$

Hence we have $A^{-1} \in C_h^{pol}G$. So $C_h^{pol}G$ is closed under holomorphic functional calculus by [15, Lemma 1.2] and [16, Lemma 3.38]. □

2.4.2 Trace Functions

In this subsubsection, we recall the trace function τ_h on $\mathbb{C}G$ corresponding to an element $h \in G$. If $h \in G^{\mathrm{pol}}$, we extend this trace to a trace $\widetilde{\tau}_h$ on $C_h^{pol}G$. Recall that a trace function on a $\mathbb{C}$-algebra $\mathcal{A}$ is a $\mathbb{C}$-linear map $\tau : \mathcal{A} \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$.

In the following, we are recalling the classical trace on $\mathbb{C}G$ corresponding to an element $h \in G$.

Definition 2.4.13. For all $h \in G$, let $\tau_h : \mathbb{C}G \to \mathbb{C}$ be defined as:

$$\tau_h(\sum_{g \in G} a_g.g) := \sum_{g \in C(h)} a_g,$$

where $C(h)$ is the conjugacy class of h.

It is clear that τ_h is a trace function on $\mathbb{C}G$ [17].

In the following, we introduce a notation so that, we can use operators as matrices.

Definition 2.4.14. Given $A \in \mathcal{B}(\ell^2 G)$, define $A(g, f) := (A\delta_f)(g)$ for all $g, f \in G$, where

$$\delta_f(k) = \begin{cases} 1 & \text{if } k = f \\ 0 & \text{otherwise .} \end{cases}$$

The following equivariance property is used in the proof that liftings

$$\widetilde{\tau}_h : C_h^{pol} G \to \mathbb{C}$$

are trace functions. It can be shown with a direct calculation.

Lemma 2.4.15. *[12] Given $A \in C_r^* G$ and $g, f, h \in G$, we have $A(g, f) = A(gh, fh)$.*

In the following, we define a lifting of the classical trace function $\tau_h : \mathbb{C}G \to \mathbb{C}$.

Definition 2.4.16. For each $h \in G^{\mathrm{pol}}$, define $\widetilde{\tau}_h : C_h^{pol} G \to \mathbb{C}$ as,

$$\widetilde{\tau}_h(A) = \sum_{g \in C(h)} A(g, e) \; .$$

In the following, we prove an inequality that we use in the proof of the Theorem 2.4.18.

Lemma 2.4.17. *[12] For all $A \in \mathcal{B}(\ell^2 G)$ and $R \in \mathbb{R}_{>0}$ we have*

$$\left(\sum_{\|g\|_w > R} |A(g, e)|^2 \right)^{\frac{1}{2}} \leq \mu_A(R) \; .$$

Proof. For all $A \in \mathcal{B}(\ell^2 G)$ and $R \in \mathbb{R}_{>0}$ we have

$$\begin{aligned}
\left(\sum_{\|g\|_w > R} |A(g,e)|^2\right)^{\frac{1}{2}} &= \left(\sum_{g \in G \setminus B_R(\{e\})} |(A\delta_e)(g)|^2\right)^{\frac{1}{2}} \\
&= \|A\delta_e\|_{G \setminus B_R(\mathrm{Supp}\ \delta_e)} \\
&\leq \mu_A(R)\|\delta_e\| \\
&= \mu_A(R).
\end{aligned}$$

□

Since we defined $\widetilde{\tau}_h$ to be a sum over the (possibly infinite) set $C(h)$, we need to prove that the sum converges. In the following, we show that the sum absolutely converges.

Theorem 2.4.18. *[12] $\widetilde{\tau}_h : C_h^{pol} G \to \mathbb{C}$ is well defined and continuous.*

Proof. Given $A \in C_h^{pol} G$, we have 2 cases:

If $h = e$, then $|\widetilde{\tau}_h(A)| = |A(e,e)| < \infty$. If $h \neq e$, then we have

$$\begin{aligned}
|\widetilde{\tau}_h(A)| &\leq \sum_{l=1}^{\infty} \sum_{g \in C_l(h)} |A(g,e)| && \text{(since } h \neq e,\ l \text{ starts from 1)} \\
&\leq \sum_{l=1}^{\infty} \sqrt{n_{h,l}} \cdot \left(\sum_{g \in C_l(h)} |A(g,e)|^2\right)^{\frac{1}{2}} && \text{(Cauchy-Schwarz inequality)} \\
&\leq \sum_{l=1}^{\infty} \sqrt{n_{h,l}} \cdot \left(\sum_{\|g\|_w > (l-\frac{1}{2})} |A(g,e)|^2\right)^{\frac{1}{2}} \\
&\leq \sum_{l=1}^{\infty} \sqrt{n_{h,l}} \cdot \mu_A\left(l - \frac{1}{2}\right) && \text{(Lemma 2.4.17)} \\
&\leq \sum_{l=1}^{\infty} \sqrt{C_h} \cdot l^{\frac{d_h}{2}} \cdot \|A\|_{\mu,h} \cdot \left(l - \frac{1}{2}\right)^{-b_h} \\
&\leq C \cdot \sqrt{C_h} \cdot \|A\|_{\mu,h} \cdot \sum_{l=1}^{\infty} l^{-2}, \text{ for some } C \in \mathbb{R}_{>0}. && \left(b_h \geq \frac{d_h}{2} + 2\right) \\
&< \infty\ .
\end{aligned}$$

Hence $\widetilde{\tau}_h : C_h^{pol} G \to \mathbb{C}$ is well-defined and continuous. □

In the following, we show indeed $\widetilde{\tau}_h : C_h^{pol}G \to \mathbb{C}$ is a trace function extending the classical trace function $\tau_h : \mathbb{C}G \to \mathbb{C}$.

Theorem 2.4.19. *[12] For all $h \in G^{\mathrm{pol}}$, $\widetilde{\tau}_h$ is a trace function on $C_h^{pol}G$ extending τ_h.*

Proof. The only nontrivial part is to show that $\forall A, B \in C_h^{pol}G$ we have $\widetilde{\tau}_h(AB) = \widetilde{\tau}_h(BA)$:

Given $A, B \in C_h^{pol}G$, we have †

$$\begin{aligned}
\widetilde{\tau}_h(AB) &= \sum_{f\in G}\sum_{g\in C(h)} A(gf^{-1}, e)B(f, e) \\
&= \sum_{f\in G}\sum_{k\in C(h)} A(f^{-1}k, e)B(f, e) && (k = fgf^{-1}) \\
&= \sum_{k\in C(h)}\sum_{f\in G} A(f^{-1}k, e)B(f, e) \\
&= \sum_{k\in C(h)}\sum_{l\in G} A(l, e)B(kl^{-1}, e) && (l = f^{-1}k) \\
&= \widetilde{\tau}_h(BA).
\end{aligned}$$

□

In the following, we show that trace functions induce homomorphism on the K_0 group.

Lemma 2. For a C^*-algebra $\mathcal{A}$ and a trace function $\tau : \mathcal{A} \to \mathbb{C}$, there is an induced homomorphism $\widetilde{\tau} : K_0(\mathcal{A}) \to \mathbb{C}$.

Proof. Given

$$p = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \in \mathcal{P}_n(\mathcal{A})$$

We define $\widetilde{\tau}([p]) = \sum_{i=1}^{n} \tau(p_{ii})$.

†We have absolute convergence in the sums (by the proof of Lemma 2.4.18). So we can change the order of summation as we want.

In order to show that $\widetilde{\tau}$ is well defined, let

$$q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1m} \\ q_{21} & q_{22} & \cdots & q_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ q_{m1} & q_{m2} & \cdots & q_{mm} \end{pmatrix} \in \mathcal{P}_m(\mathcal{A})$$

such that $p \sim_0 q$. Then there is a matrix

$$v = \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nm} \end{pmatrix} \in M_{n,m}(\mathcal{A})$$

such that $p = vv^*$ and $q = v^*v$. Then

$$\begin{aligned}
\widetilde{\tau}([p]) &= \sum_{i=1}^{n} \tau(p_{ii}) \\
&= \sum_{i=1}^{n} \tau(\sum_{j=1}^{n'} v_{ij} v_{ji}^*) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n'} \tau(v_{ij} v_{ji}^*) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n'} \tau(v_{ji}^* v_{ij}) \\
&= \sum_{j=1}^{n'} \sum_{i=1}^{n} \tau(v_{ji}^* v_{ij}) \\
&= \sum_{j=1}^{n'} \tau(\sum_{i=1}^{n} v_{ji}^* v_{ij}) \\
&= \sum_{j=1}^{n'} \tau(q_{jj}) \\
&= \widetilde{\tau}([q])
\end{aligned}$$

Thus $\widetilde{\tau}$ is a well defined group homomorphism. $\square$

3. MAIN RESULT*

In this section, we introduce numerical invariants which are the bounds for the ranks of the finite parts of the K_0 groups. We state and prove our main result at the end of the section.

3.1 Numerical Invariants

In this subsection, for a finitely generated group G, we define two numbers $\mathcal{F}_G$ and $\mathcal{F}_G^{\text{pol}}$ which are the upper and lower bounds for the ranks of finite parts in our main result.

For a finitely generated group G, recall that G^{fin} is the set of elements in G with finite order. In other words, $G^{\text{fin}} = \{g \in G : g^a = e$ for some natural number $a\}$, also recall that G^{pol} is the set of elements with a polynomially growing conjugacy class.

In the following, we define a relation on the set G^{fin}.

Definition 3.1.1. For $g, h \in G$ we define $g \sim_{\text{fin}} h$ if $\text{order}(g) = \text{order}(h)$ and there is a natural number a such that g^a is a conjugate of h.

In the following, we show that the relation defined above is an equivalence relation.

Theorem 3.1.2. *[12] The relation $\sim_{\text{fin}}$ is an equivalence relation on G^{fin}.*

Proof. For all $d \in \mathbb{N}$ let G_d^{fin} be the elements in G^{fin} with order d.

For all $g \in G_d^{\text{fin}},\ we\ have\ g^1 = ege^{-1} \in C(g),\ so\ g \sim_{\text{fin}} g.$

Now for all $g, h \in G_d^{\text{fin}}$, if $g \sim_{\text{fin}} h$, then $\exists\ a \in \mathbb{N}, f \in G$ such that $g^a = fhf^{-1}$. Since $\text{order}(g^a) = \text{order}(fhf^{-1}) = \text{order}(h) = d$ we get, $\gcd(a, d) = 1$. So, $\exists\ b \in \mathbb{N}$ such that $ab \equiv 1(mod\ d)$. Then $h^b = f^{-1}fh^bf^{-1}f = f^{-1}(fhf^{-1})^bf = f^{-1}g^{ab}f = f^{-1}gf \in C(g)$. Hence, $h \sim_{\text{fin}} g$.

Now $\forall f, g, h \in G_d^{\text{fin}}$, if $f \sim_{\text{fin}} g$ and $g \sim_{\text{fin}} h$, then $\exists\ a, b \in \mathbb{N}, \exists\ k, l \in G$ such that $f^a = kgk^{-1}$ and $g^b = lhl^{-1}$. Hence, we have $f^{ab} = kg^bk^{-1} = (kl)h(kl)^{-1} \in C(h)$. Thus, $f \sim_{\text{fin}} h$. □

*Portions of this section are reprinted with permission from [12].

In the following, we define the numerical invariants which are the bounds we use in our main theorem.

Definition 3.1.3. [12] For a finitely generated group G, define

$$\mathcal{F}_G = |G^{\mathrm{fin}}/\sim_{\mathrm{fin}}|$$

and

$$\mathcal{F}_G^{\mathrm{pol}} = |(G^{\mathrm{fin}} \cap G^{\mathrm{pol}})/\sim_{\mathrm{fin}}|.$$

3.2 Main Result

In this subsection we state our main result in which we give a lower and an upper bound for the ranks of the finite parts of the K_0 groups of the maximal and reduced group C^*-algebras.

Theorem 3.2.1. *[12] Let G be a finitely generated group. We have*

$$\mathcal{F}_G^{\mathrm{pol}} \leq \mathrm{rank}(K_0^{\mathrm{fin}}(C_r^*G)) \leq \mathrm{rank}(K_0^{\mathrm{fin}}(C^*G)) \leq \mathcal{F}_G$$

and for the assembly map $\mu : K_0^G(EG) \to K_0(C^*G)$, we have

$$\mathrm{Im}(\mu) \cap K_0^{\mathrm{fin,pol}}(C^*G) = \{0\},$$

where $K_0^{\mathrm{fin,pol}}(C^*G)$ is the subgroup of $K_0(C^*G)$ generated by the set

$$\{[p_g] \ : \ g \in G^{\mathrm{fin}} \cap G^{\mathrm{pol}}\}.$$

Note that this result follows from the injectivity part of the Baum-Connes conjecture [1].

Gong [18] finds a lower bound for the rank of $K_0^{fin}(C_r^*G)$ for the groups with property (RD) and conjugacy classes having poynomial growth. In our results, we don't require property (RD) and also improve the lower bound.

3.3 Proof of the Main Result

In this subsection, we give the proof of Theorem 3.2.1. In order to give the proof, we need a framework theorem. In the following, we state and prove the framework theorem we use to prove the main theorem.

Theorem 3.3.1. *[12] Let $S \subseteq G^{\text{fin}}$. If there exists a smooth subalgebra $\mathcal{A}$ of C_r^*G containing $\mathbb{C}G$ and if $\forall h \in S$ there exists a trace function*

$$\widetilde{\tau}_h : \mathcal{A} \to \mathbb{C}$$

extending the trace function $\tau_h : \mathbb{C}G \to \mathbb{C}$, then we have

$$\operatorname{rank}(K_0^{\text{fin}}(C_r^*G)) \geq \operatorname{rank}(K_0^S(C_r^*G)) \geq |S/_{\sim_{\text{fin}}}|$$

and for the assembly map $\mu : K_0^G(EG) \to K_0(C^*G)$, we have

$$\operatorname{Im} \mu \cap K_0^S(C^*G) = \{0\},$$

where $K_0^S(C^*G)$ is the subgroup of $K_0(C^*G)$ generated by the set $\{[p_g] : g \in S\}$ and EG is the universal cover of the classifying space BG.

Proof. Since $\mathcal{A}$ is a smooth and dense subalgebra of C_r^*G, we have

$$K_0(\mathcal{A}) \cong K_0(C_r^*G),$$

where the isomorphism is induced by the inclusion map

$$i : \mathcal{A} \to C_r^*G \ .$$

Since the finite parts of $K_0(\mathcal{A})$ and $K_0(C_r^*G)$ are coming from $\mathbb{C}G$, we have

$$K_0^{fin}(\mathcal{A}) \cong K_0^{fin}(C_r^*G).$$

Hence, for the first part of the theorem, it suffices to show that

$$\mathrm{rank}(K_0^{fin}(\mathcal{A})) \geq |S/_{\sim_{\mathrm{fin}}}|.$$

Let $K_0^S(\mathcal{A})$ be the subgroup of $K_0^{fin}(\mathcal{A})$ generated by the set $\{[p_g] : g \in S\}$. Thus, it suffices to show that $\mathrm{rank}(K_0^S(\mathcal{A})) \geq |S/_{\sim_{\mathrm{fin}}}|$.

Let $\{s_1, \cdots, s_n\}$ be an arbitrary subset of S such that, we have $s_i \nsim_{\mathrm{fin}} s_j$ for $i \neq j$. We are going to show that, the subgroup of $K_0^S(\mathcal{A})$ generated by the set $\{[p_{s_1}], \cdots, [p_{s_n}]\}$ has rank n. Therefore, we are going to conclude that

$$\mathrm{rank}(K_0^S(\mathcal{A})) \geq |S/_{\sim_{\mathrm{fin}}}|.$$

For all $i \in \{1, 2, \cdots, n\}$ *let* $d_i = \mathrm{order}(s_i)$ and assume $d_1 \leq d_2 \leq \cdots \leq d_n$. We have all the traces $\widetilde{\tau}_{s_i} : \mathcal{A} \to \mathbb{C}$ defined. This gives us the homomorphisms (with abuse of notation)

$$\widetilde{\tau}_{s_i} : K_0^S(\mathcal{A}) \to \mathbb{C}.$$

Now define

$$\mathcal{M}_n = \begin{pmatrix} \widetilde{\tau}_{s_1}([p_{s_1}]) & \widetilde{\tau}_{s_1}([p_{s_2}]) & \cdots & \widetilde{\tau}_{s_1}([p_{s_n}]) \\ \widetilde{\tau}_{s_2}([p_{s_1}]) & \widetilde{\tau}_{s_2}([p_{s_2}]) & \cdots & \widetilde{\tau}_{s_2}([p_{s_n}]) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\tau}_{s_n}([p_{s_1}]) & \widetilde{\tau}_{s_n}([p_{s_2}]) & \cdots & \widetilde{\tau}_{s_n}([p_{s_n}]) \end{pmatrix},$$

where $p_{s_j} = \frac{1+s_j+\ldots+s_j^{d_j-1}}{d_j} \in \mathbb{C}G \subseteq \mathcal{A}$ and $[p_{s_j}]$ shows the class in $K_0^S(\mathcal{A})$ represented by the projection p_{s_j}. Now $\forall i, j \in \{1, 2, ..., n\}$ with $i > j$, there are 2 cases:

Case 1 ($d_i > d_j$)

In this case, we have

$$\widetilde{\tau}_{s_i}([p_{s_j}]) = \tau_{s_i}(p_{s_j}) = \tau_{s_i}\left(\frac{1 + s_j + ... + s_j^{d_j - 1}}{d_j}\right)$$

and since $\forall a \in \mathbb{N}$ $\operatorname{order}(s_j^a) \leq \operatorname{order}(s_j) = d_j < d_i = \operatorname{order}(s_i)$, we have $\forall a \in \mathbb{N}$ $s_j^a \notin C(s_i)$ (all elements from $C(s_i)$ have order d_i). Thus, $\widetilde{\tau}_{s_i}([p_{s_j}]) = 0$.

Case 2 $(d_i = d_j)$ **and** $s_j \nsim_{\text{fin}} s_i$

In this case, we have $\forall a \in \mathbb{N}$ $s_j^a \notin C(s_i)$ by definition of $\sim_{\text{fin}}$. So $\widetilde{\tau}_{s_i}([p_{s_j}]) = 0$.

Hence, $\mathcal{M}_n$ is an upper triangular matrix.

Now $\forall i \in \{1, 2, ..., n\}$, we have $s_i \in C(s_i)$ so,

$$\begin{aligned}
\widetilde{\tau}_{s_i}([p_{s_i}]) &= \tau_{s_i}(p_{s_i}) \\
&= \tau_{s_i}\left(\frac{1 + s_i + ... + s_i^{d_i - 1}}{d_i}\right) \\
&\geq \frac{1}{d_i} \ .
\end{aligned}$$

Thus, the elements in the diagonal of $\mathcal{M}_n$ are non-zero. Hence, $det(\mathcal{M}_n) \neq 0$. So $\mathcal{M}_n$ has full rank. Thus, in $K_0^S(\mathcal{A})$, the elements $[p_{s_1}], ..., [p_{s_n}]$ are $\mathbb{Z}$-linearly independent. Therefore, $\operatorname{rank}(K_0^S(\mathcal{A})) = n$. Thus, we get

$$\operatorname{rank}(K_0^{fin}(\mathcal{A})) \geq |S/_{\sim_{\text{fin}}}|.$$

Now, let's make some preliminary definitions for the proof of the second part of the Theorem 3.3.1:

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. We define $\mathcal{S}_p := \{T \in \mathcal{B}(\mathcal{H}) \mid tr((T^*T)^{\frac{p}{2}}) < \infty\}$, where $tr(P) := \sum_{n\in\mathbb{N}} \langle Pe_n, e_n\rangle$ for an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and for a bounded linear operator $P \in \mathcal{B}(\mathcal{H})$. We remark that, trace does not depend on the particular choice of an orthonormal basis. We call $\mathcal{S}_p$ the ring of Schatten p-class operators on an infinite dimensional and separable Hilbert space. Now define $\mathcal{S} := \cup_{p=1}^{\infty} \mathcal{S}_p$. The ring $\mathcal{S}$ is called the ring of Schatten class operators. Let $\mathcal{S}G$ be the group algebra over the ring $\mathcal{S}$ [19]. Let $j : \mathbb{C}G \to \mathcal{S}G$ be the inclusion homomorphism defined by:

$$j(a) = p_0 a$$

for all $a \in \mathbb{C}G$, where p_0 is a rank one projection in $\mathcal{S}$.

In the following, we show that nonzero elements in the finite part of $K_0(C^*G)$ generated by the set $\{[p_g] : g \in S\}$ are not in the image of the assembly map $\mu : K_0^G(EG) \to K_0(C^*G)$, where EG is the universal space for proper and free G-action. In the proof, we use the n-cocycle $\tau_g^{(n)}$ on $\mathcal{S}_m G$ introduced in [2], which gives in some sense the extension of the classical trace τ_g. So we have a commutative diagram

$$\begin{tikzcd} K_0(\mathcal{S}_m G) \arrow[r, "\psi"] \arrow[d, "(\tau_g^{(n)})_*"'] & K_0(C^*G) \arrow[dl, "\widetilde{\tau}_g"] \\ \mathbb{C} & \end{tikzcd}$$

where (with abuse of notation) $\widetilde{\tau}_g : K_0(C^*G) \to \mathbb{C}$ is the pullback of the homomorphism $\widetilde{\tau}_g : K_0(C_r^*G) \to \mathbb{C}$. Recall that $K_0(\mathcal{A}) \cong K_0(C_r^*G)$.

Assume there exists a non-zero $z \in \operatorname{Im}(\mu) \cap K_0^S(C^*G)$. Then $z = \sum_{i=1}^{s} c_i \cdot [p_{g_i}]$ for some pairwise non-equivalent $g_1, \cdots, g_s \in S$ and $c_1, \cdots, c_s \in \mathbb{Z} \setminus \{0\}$. For all $i \in \{1, \cdots, s\}$ let $d_i = \operatorname{order}(g_i)$. Without loss of generality, we can assume $d_1 \leq \cdots \leq d_s$. Now let $g = g_s$.

Now let $z' = \sum_{i=1}^{s} c_i \cdot [j(p_{g_i})]$. We have $z' \in K_0(\mathcal{S}_m G)$ for some $m \in \mathbb{N}$. Then we get $z' \in \operatorname{Im}(A)$, where

$$A : H_0^{OrG}(EG, \mathbb{K}(\mathcal{S}_m)^{-\infty}) \to K_0(\mathcal{S}_m G)$$

is the assembly map.

Let $n = 2k$ be the smallest even number greater than or equal to m. Define an n-cocycle $\tau_g^{(n)}$ on $\mathcal{S}_m G$ by:

$$\tau_g^{(n)}(a_0, a_1, \cdots, a_n) := \sum_{\gamma \in C(g)} tr(\gamma^{-1} a_0 a_1 \cdots a_n)$$

for all $a_i \in \mathcal{S}_m G$, where $tr : \mathcal{S}_1 G \to \mathbb{C}$, is the trace defined by:

$$tr(\sum_{\gamma \in G} b_\gamma \gamma) := trace(b_e) \ .$$

Since $\tau_g^{(n)}$ is an n-cocycle, it induces a homomorphism

$$(\tau_g^{(n)})_* : K_0(\mathcal{S}_m G) \to \mathbb{C}.$$

It is shown in [2] that $(\tau_g^{(n)})_*([j(p)]) = \tau_g(p)$ for all projections $p \in \mathbb{C}G$ and $(\tau_g^{(n)})_*(z') = 0$. So we have the commutative diagram

$$\begin{tikzcd}
K_0(\mathcal{S}_m G) \arrow[r, "\psi"] \arrow[d, "(\tau_g^{(n)})_*"'] & K_0(C^* G) \arrow[dl, "\widetilde{\tau}_g"] \\
\mathbb{C} &
\end{tikzcd}$$

where (with abuse of notation) $\widetilde{\tau}_g : K_0(C^* G) \to \mathbb{C}$ is the pullback of the homomorphism $\widetilde{\tau}_g : K_0(C_r^* G) \to \mathbb{C}$. We have $\psi(z') = z$. Hence, we get

$$\widetilde{\tau}_g(z) = \widetilde{\tau}_g(\psi(z')) = (\tau_g^{(n)})_*(z') = 0.$$

However, we have $\widetilde{\tau}_g(z) = \widetilde{\tau}_g(\sum_{i=1}^s c_i \cdot [p_{g_i}]) = \sum_{i=1}^s c_i \cdot \widetilde{\tau}_g([p_{g_i}]) = k \cdot \frac{c_s}{d_s}$ for some $k \in \mathbb{N}$. Thus, $\widetilde{\tau}_g(z) \neq 0$. Contradiction shows that $\mathrm{Im}(\mu) \cap K_0^S(C^* G) = \{0\}$. $\square$

In the following, we prove our main theorem.

Proof of Theorem 3.2.1. Since, for all $g, h \in G^{\text{fin}}$, we have

$$g \sim_{\text{fin}} h \implies [p_g] = [p_h] \in K_0^{\text{fin}}(C^*G),$$

we get $\operatorname{rank}(K_0^{\text{fin}}(C^*G)) \leq \mathcal{F}_G$. Using the surjection $K_0^{\text{fin}}(C^*G) \twoheadrightarrow K_0^{\text{fin}}(C_r^*G)$, we can conclude that $\operatorname{rank}(K_0^{\text{fin}}(C_r^*G)) \leq \operatorname{rank}(K_0^{\text{fin}}(C^*G))$.

For the rest, it suffices to prove that $S = G^{\text{pol}} \cap G^{\text{fin}}$ and $\mathcal{A} = C_S^{pol}G := \bigcap_{h \in S} C_h^{pol}G$ satisfies the hypotheses of the Theorem 3.3.1.

Since $C_h^{pol}G's$ are smooth subalgebras of C_r^*G containing $\mathbb{C}G$, we get $C_S^{pol}G$ is a smooth subalgebra of C_r^*G containing $\mathbb{C}G$.

Since $G^{\text{pol}} \cap G^{\text{fin}} \subseteq G^{\text{pol}}$, for all $h \in G^{\text{pol}} \cap G^{\text{fin}}$, $\tau_h : \mathbb{C}G \to \mathbb{C}$ has a lift

$$\widetilde{\tau}_h : C_h^{pol}G \to \mathbb{C}.$$

Therefore, for all $h \in S$, the trace function $\tau_h : \mathbb{C}G \to \mathbb{C}$ has a lift

$$\widetilde{\tau}_h : C_S^{pol}G \to \mathbb{C},$$

which is also a trace function.

Hence, we get $\operatorname{rank}(K_0^{\text{fin}}(C_r^*G)) \geq |S/_{\sim_{\text{fin}}}| = |(G^{\text{pol}} \cap G^{\text{fin}})/_{\sim_{\text{fin}}}| = \mathcal{F}_G^{\text{pol}}$. For the assembly map $\mu : K_0^G(EG) \to K_0(C^*G)$, we have $\operatorname{Im}\mu \cap K_0^{\text{fin,pol}}(C^*G) = \{0\}$. $\square$

Remark 3.3.2. By the proof above we also get $\operatorname{rank}(K_0^{\text{fin,pol}}(C^*G)) \geq |S/_{\sim_{\text{fin}}}| = \mathcal{F}_G^{\text{pol}}$.

4. APPLICATIONS*

In this section, we combine the results from Weinberger and Yu [2] and Theorem 3.2.1 to derive lower bounds for the ranks of the structure group and the group of positive scalar curvature metrics of manifolds.

4.1 Application About the Structure Group

Given a compact oriented manifold M, we define the structure group $S(M)$ of M to be the abelian group generated by the equivalence classes of elements of the form (f, M'), where M' is a compact oriented manifold and $f : M' \to M$ is an orientation preserving homotopy equivalence. We say (f_1, M_1) is equivalent to (f_2, M_2) if there exists an h-cobordism $(W; M_1, M_2)$ and a homotopy equivalence $F : W \to M$ such that restrictions of F to M_1 and M_2 gives f_1 and f_2 respectively [20, Definition 1.14].

Weinberger, Xie and Yu [21] use the higher rho invariant to study the structure group $S(M)$.

We have the following result about the structure group $S(M)$ of a compact oriented manifold M from Weinberger and Yu.

Theorem 4.1.1. *[2] Let M be a compact oriented manifold with dimension $4k-1$ $(k > 1)$. Suppose $\pi_1(M) = G$ and $g_1, \cdots, g_n$ be finite order elements in G such that $g_i \neq e$ for all i and $\{[p_{g_1}], \cdots, [p_{g_n}]\}$ generates an abelian group of $K_0(C^*G)$ with rank n. Suppose that any nonzero element in the abelian subgroup of $K_0(C^*G)$ generated by $\{[p_{g_1}], \cdots, [p_{g_n}]\}$ is not in the image of the map $\mu : K_0^G(EG) \to K_0(C^*G)$, then the rank of the structure group $S(M)$ is greater than or equal to n.*

Now we combine the previous result about $S(M)$ with Theorem 3.2.1, where the lower bound is in terms of $\mathcal{F}_G^{pol}$.

Corollary 4.1.2. *[12] For a compact oriented manifold M with dimension $4k-1$ $(k > 1)$, the rank of the structure group $S(M)$ is greater than or equal to $\mathcal{F}_G^{pol} - 1$, where $G = \pi_1(M)$.*

*Portions of this section are reprinted with permission from [12].

Proof. We have $\operatorname{rank}(K_0^{fin,pol}(C^*G)) \geq \mathcal{F}_G^{pol}$ by the proof of Theorem 3.2.1 and, we have $\operatorname{Im}(\mu) \cap K_0^{fin,pol}(C^*G) = \{0\}$. Since we have $e \in G^{\mathrm{fin}} \cap G^{\mathrm{pol}}$, we get the rank of the structure group $S(M)$ is greater than or equal to $\mathcal{F}_G^{pol} - 1$. □

4.2 Application About the Group of Positive Scalar Curvature Metrics

In this subsection, we apply our main result to the results of Weinberger and Yu [2] to derive a concrete lower bound for the rank of the group of positive scalar curvature metrics of a compact smooth spin manifold M in terms of its fundamental group $G = \pi_1(M)$.

Let $r_{fin}(G)$ be the rank of the abelian group $K_0^{fin}(C^*G)$ generated by $[p_g]$ for all finite order elements $g \in G$. Here g is allowed to be the identity element e. So we have $r_{fin}(G) = \operatorname{rank}(K_0^{fin}(C^*G)) \geq \mathcal{F}_G^{pol}$.

If a compact smooth spin manifold M has a positive scalar curvature metric and the dimension of M is greater than or equal to 5, then we define (roughly) $P(M)$ to be the abelian group of equivalence classes of all positive scalar curvature metrics on M. For a more precise definition, we refer to [2, Section 4].

We have the following result about the group $P(M)$ from Weinberger and Yu.

Theorem 4.2.1. *[2]*

1. *Let M be a compact smooth spin manifold with a positive scalar curvature metric and dimension $2k-1$ $(k > 2)$. The rank of the abelian group $P(M)$ is greater than or equal to $r_{fin}(G) - 1$.*

2. *Let M be a compact smooth spin manifold with a positive scalar curvature metric and dimension $4k-1$ $(k > 1)$. The rank of the abelian group $P(M)$ is greater than or equal to $r_{fin}(G)$.*

In the following, we combine the previous result about $P(M)$ with Theorem 3.2.1. The lower bounds are in terms of $\mathcal{F}_G^{pol}$.

Corollary 4.2.2. *[12] Let M be a compact smooth spin manifold with a positive scalar curvature metric and let $G = \pi_1(M)$.*

1. *If M has dimension $2k-1$ $(k > 2)$, then the rank of the abelian group $P(M)$ is greater than or equal to $\mathcal{F}_G^{pol} - 1$.*
2. *If M has dimension $4k-1$ $(k > 1)$, then the rank of the abelian group $P(M)$ is greater than or equal to $\mathcal{F}_G^{pol}$.*

Proof. We have $r_{fin}(G) \geq \mathcal{F}_G^{pol}$. $\square$

5. POLYNOMIALLY FULL GROUPS*

In this section, we define the class of polynomially full groups. We show that finitely generated subgroups, products, finite extensions and images with finite kernels of polynomially full groups are also polynomially full. For a polynomially full group G, we show that

$$K_0^{fin}(C_r^*G) \cong K_0^{fin}(C^*G) \cong \bigoplus_{i=1}^{\mathcal{F}_G} \mathbb{Z}.$$

The class of polynomially full groups includes trivially all finite groups and finitely generated torsion-free groups. We show that it also includes all finitely generated virtually nilpotent groups. At the end of the section, we derive formulas for the number $\mathcal{F}_G$, where G is finitely generated abelian group, dihedral group, or symmetric group.

For a finitely generated group G with a finite generating set S, we denote the word-length norm by $\|.\|$, $\|.\|_S$, or $\|.\|_G$. For $g \in G$, we denote the conjugacy class of g in G by $C^G(g)$. We denote the set of elements in the conjugacy class of g with length (with respect to S) l by $C_l^G(g)$.

5.1 Definition of the Poynomially Full Groups

In this subsection, we define the class of polynomially full groups and derive some important property of these groups. Recall that G^{fin} is the set of torsion elements and G^{pol} is the set of elements with a polynomially growing conjugacy class.

In the following, we give two equivalent conditions for a group. We use these conditions to define the class of polynomially full groups.

Proposition 5.1.1. *[12] For a finitely generated group G the following are equivalent:*

1. $G^{\mathrm{fin}} \subseteq G^{\mathrm{pol}}$.
2. *For all* $g \in G^{\mathrm{fin}}$ *there exists* $h \in G^{\mathrm{fin}} \cap G^{\mathrm{pol}}$ *such that* $g \sim_{\mathrm{fin}} h$ *(i.e.* $G^{\mathrm{fin}}/\sim_{\mathrm{fin}} = (G^{\mathrm{fin}} \cap G^{\mathrm{pol}})/\sim_{\mathrm{fin}}$*).*

*Portions of this section are reprinted with permission from [12].

Proof. (1) $\implies$ (2) : Obvious.

(2) $\implies$ (1) : Given $g \in G^{\text{fin}}$, there exists $h \in G^{\text{fin}} \cap G^{\text{pol}}$ such that $g \sim_{\text{fin}} h$. So there exist $a, b \in \mathbb{N}$ and $f \in G$ such that $g^a = fhf^{-1}$ and $h^b \in C^G(g)$. Define $A_l = \{\alpha^a \ : \ \alpha \in C_l^G(g)\}$. Define $\mathrm{i} : C_l^G(g) \to A_l$ by

$$\mathrm{i}(\alpha) = \alpha^a$$

and $\mathrm{j} : A_l \to C_l^G(g)$ by

$$\mathrm{j}(\beta) = \beta^b.$$

It is easy to see that, $\mathrm{j} \circ \mathrm{i} = \mathrm{id}_{C_l^G(g)}$ and $\mathrm{i} \circ \mathrm{j} = \mathrm{id}_{A_l}$. Hence, we have

$$|C_l^G(g)| = |A_l|.$$

Now, let $B_l = \{\beta \in C^G(h) \ : \ \|\beta\| \leq a \cdot l\}$. We show that $A_l \subseteq B_l$:

Given $\omega \in A_l$, there exists $\alpha \in C^G(g)$ with $\|\alpha\| = l$ and $\omega = \alpha^a$. Since $\alpha \in C^G(g)$, there exists $\gamma \in G$ such that $\alpha = \gamma g \gamma^{-1}$. So $\omega = \alpha^a = \gamma g^a \gamma^{-1} = \gamma f h f^{-1} \gamma^{-1}$. Hence, $\omega \in C^G(h)$ with $\|\omega\| = \|\alpha^a\| \leq a \cdot \|\alpha\| = a \cdot l$. Thus, $\omega \in B_l$. Therefore, we have $A_l \subseteq B_l$.

So we get $|C_l^G(g)| = |A_l| \leq |B_l|$. Since $h \in G^{\text{pol}}$, $|B_l|$ is bounded from above by a polynomial of l. Thus, we get $g \in G^{\text{pol}}$. ∎

Definition 5.1.2. Let G be a finitely generated group. We say that G is polynomially full, if it satisfies the conditions from Proposition 5.1.1.

Obviously finite groups and finitely generated torsion-free groups are polynomially full.

The following result is the motivation behind the definition of polynomially full groups.

Theorem 5.1.3. *[12] For a polynomially full group G we have*

$$K_0^{fin}(C_r^* G) \cong K_0^{fin}(C^* G) \cong \bigoplus_{i=1}^{\mathcal{F}_G} \mathbb{Z}.$$

Proof. Let G be a polynomially full group. We have

$$\begin{aligned}\mathcal{F}_G &= |G^{\text{fin}}/_{\sim_{\text{fin}}}| \\ &= |(G^{\text{fin}} \cap G^{\text{pol}})/_{\sim_{\text{fin}}}| \\ &= \mathcal{F}_G^{pol}\end{aligned}$$

So we have $\bigoplus_{i=1}^{\mathcal{F}_G} \mathbb{Z} = \bigoplus_{i=1}^{\mathcal{F}_G^{pol}} \mathbb{Z}$. Let $F = (G^{\text{fin}} \cap G^{\text{pol}})/_{\sim_{\text{fin}}}$. Hence, we have $\bigoplus_{i=1}^{\mathcal{F}_G^{pol}} \mathbb{Z} \cong \bigoplus_{[g]\in F} \mathbb{Z}$. Now, define

$$\phi : \bigoplus_{[g]\in F} \mathbb{Z} \to K_0^{fin}(C_r^* G)$$

as

$$\phi(\delta_{[g]}) = [p_g],$$

where $\delta_{[g]}$ is the canonical basis element corresponding to the $[g]$-component. Since $g \sim_{\text{fin}} h$ implies $[p_g] = [p_h]$, ϕ is a well defined homomorphism. Recall that, $K_0^{fin}(C_r^* G)$ is generated by the set $\{[p_g] : g \in G^{\text{fin}}\}$. Since G is polynomially full, we have

$$\{[p_g] : g \in G^{\text{fin}}\} = \{[p_g] : g \in G^{\text{fin}} \cap G^{\text{pol}}\}.$$

Hence, ϕ is surjective.

On the other hand, by the proof of the Theorem 3.3.1, $\phi(\delta_{[g]}) = [p_g]'s$ are $\mathbb{Z}$-linearly independent. Thus, ϕ is injective. So we have $\bigoplus_{[g]\in F} \mathbb{Z} \cong K_0^{fin}(C_r^* G)$. Therefore, we get $K_0^{fin}(C_r^* G) \cong \bigoplus_{i=1}^{\mathcal{F}_G} \mathbb{Z}$. The isomorphism $K_0^{fin}(C^* G) \cong \bigoplus_{i=1}^{\mathcal{F}_G} \mathbb{Z}$ can be shown similarly. □

5.2 Closure Properties

In this subsection, we study the closure properties of the class of polynomially full groups and show that all finitely generated virtually nilpotent groups are polynomially full. Recall that a group is called virtually nilpotent if it contains a nilpotent subgroup with finite index.

In the following, we show that finite extensions and images of polynomially full groups under homomorphisms with finite kernels are also polynomially full.

Proposition 5.2.1. *[12] Let F be a finite group and let G, H be finitely generated groups. If we have a short exact sequence*

$$1 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1,$$

then G is polynomially full if and only if H is polynomially full. In this case, we have $\mathcal{F}_H \leq \mathcal{F}_G$.

Proof. Without loss of generality, assume that the generating set of H is the image of the generating set of G under β.

Assume G is polynomially full. Given $h \in H^{fin}$, since β is onto, there exists $g \in G$ with $\beta(g) = h$. Since F is finite, we get $g \in G^{\text{fin}}$. Now, since G is polynomially full, $\exists g' \in G^{\text{fin}} \cap G^{\text{pol}}$ such that $g \sim_{\text{fin}} g'$. Now, let $h' = \beta(g') \in H$. Since $g' \in G^{\text{fin}}$, we have $h' \in H^{fin}$. Since $g \sim_{\text{fin}} g'$, we have $h \sim_{\text{fin}} h'$. Now, since F is finite, there exists $R \in \mathbb{N}$ such that $\ker\beta = \operatorname{Im}\alpha \subseteq \mathrm{B}_R(e_G)$, where $\mathrm{B}_R(e_G)$ is the closed ball around the identity element of G with radius R.

Since, $\cup_{l=1}^{N} C_l^H(h') \subseteq \beta(\cup_{l=1}^{N+R} C_l^G(g'))$, we have $|\cup_{l=1}^{N} C_l^H(h')| \leq |\cup_{l=1}^{N+R} C_l^G(g')|$, and right hand side is bounded by some polynomial of $N + R$ (hence by a polynomial of N). So we get $h' \in H^{pol}$. Hence, $h' \in H^{fin} \cap H^{pol}$ with $h' \sim_{\text{fin}} h$. Therefore, H is polynomially full.

For the converse, assume H is polynomially full. Given $g \in G^{\text{fin}}$, let $h = \beta(g) \in H^{fin}$. Since H is polynomially full, we have $h \in H^{pol}$. Now we have

$$\beta(\cup_{i=0}^{l} C_i^G(g)) \subseteq \cup_{i=0}^{l} C_i^H(h).$$

Hence, we get $|\cup_{i=0}^{l} C_i^G(g)| \leq |F| \cdot |\beta(\cup_{i=0}^{l} C_i^G(g))| \leq |F| \cdot |\cup_{i=0}^{l} C_i^H(h)|$. Since $h \in H^{pol}$ and F is finite, right hand side is bounded by a polynomial of l. Thus, we get $g \in G^{\text{pol}}$. Therefore, G is polynomially full.

Now, given $g_1, g_2 \in G^{\text{fin}}$, $g_1 \sim_{\text{fin}} g_2$ (in G) implies $\beta(g_1) \sim_{\text{fin}} \beta(g_2)$ (in H). Thus, we have $\mathcal{F}_H = |H^{fin}/_{\sim_{\text{fin}}}| = |\beta(G^{\text{fin}})/_{\sim_{\text{fin}}}| \leq |G^{\text{fin}}/_{\sim_{\text{fin}}}| = \mathcal{F}_G$. $\square$

In the following, we prove that the property of being polynomially full is inherited to finitely generated subgroups.

Proposition 5.2.2. *[12] Let G be a finitely generated group. Let H be a finitely generated subgroup of G. If G is polynomially full, then H is also polynomially full.*

Proof. Let S and T be finite generating sets of G and H respectively. Without loss of generality, we can assume that $T \subseteq S$.

Now, given $h \in H^{fin}$, we have $h \in G^{\mathrm{fin}}$. Since G is polynomially full, we have $h \in G^{\mathrm{pol}}$.

It is easy to see that $C^H(h) \subseteq C^G(h)$. Now, for all $f \in H$, we have $\|f\|_T \geq \|f\|_S$. Hence, $C_l^H(h) \subseteq \cup_{i=0}^{l} C_i^G(h)$. Thus, we have $|C_l^H(h)| \leq |\cup_{i=0}^{l} C_i^G(h)|$. Since $h \in G^{\mathrm{pol}}$, right hand side is bounded by a polynomial of l. Therefore, we get $h \in H^{pol}$. Hence, H is polynomially full. $\square$

In the following, we show that the class of polynomially full groups is closed under taking direct products.

Proposition 5.2.3. *[12] Let G and H be finitely generated groups. Then G and H are polynomially full if and only if $G \times H$ is polynomially full.*

Proof. Let S and T be finite generating sets for G and H respectively. Then,

$$W := S \times \{e_H\} \cup \{e_G\} \times T$$

is a finite generating set for $G \times H$. Let $\|.\|_G,\ \|.\|_H$, and $\|.\|_{(G\times H)}$ be the word-length norms on G, H, and $G \times H$ respectively, corresponding to the generating sets S, T, and W respectively.

Assume G and H are polynomially full. Given $(g, h) \in (G \times H)^{fin}$, we have $g \in G^{\mathrm{fin}}$ and $h \in H^{fin}$. Since G and H are polynomially full, we have $g \in G^{\mathrm{pol}}$ and $h \in H^{pol}$. It is not hard to see that $\|(g', h')\|_{(G\times H)} = \|g'\|_G + \|h'\|_H$ for all $g' \in G$ and $h' \in H$. So we have

$$|C_n^{G\times H}((g,h))| = \sum_{i=0}^{n} |C_i^G(g)| \cdot |C_{n-i}^H(h)|.$$

All the terms in the sum are bounded by polynomials of n. Thus, the sum is bounded by a polynomial of n. Hence, $(g, h) \in (G \times H)^{pol}$. Therefore $G \times H$ is polynomially full.

Converse follows from Proposition 5.2.2. $\square$

In the following, we give a sufficient condition for a group to be polynomially full. Recall that a subset of a group is said to grow polynomially if the number of elements in the intersection of the subset with the closed ball of radius l centered around the identity element is bounded by a fixed polynomial of l.

Lemma 5.2.4. *[12] Let G be a finitely generated group. If G^{fin} grows polynomially, then G is polynomially full.*

Proof. For all $g \in G^{\mathrm{fin}}$ we have $C^G(g) \subseteq G^{\mathrm{fin}}$. Since G^{fin} grows polynomially, $C^G(g)$ also grows polynomially. Hence, $g \in G^{\mathrm{pol}}$. Therefore, G is polynomially full. $\square$

Wolf [22, Theorem 3.11] showed that for finitely generated group Σ and a subgroup Γ of finite index, we have that

1. Γ is finitely generated, and

2. if Γ has polynomial growth, then Σ also has polynomial growth.

He also showed in [22, Theorem 3.2] that, if Γ is a finitely generated nilpotent group, then Γ has polynomial growth.

Gromov [23] showed that if a finitely generated group Γ has polynomial growth, then it is virtually nilpotent. Recall that a group is called virtually nilpotent, if it contains a nilpotent subgroup of finite index.

In the following, we show that the class of polynomially full groups includes finitely generated virtually nilpotent groups.

Corollary 5.2.5. *[12] Let G be a finitely generated group. If G is virtually nilpotent, then G is polynomially full.*

Proof. Let H be a nilpotent subgroup of G with finite index. By [22, Theorem 3.11], H is also finitely generated. So by [22, Theorem 3.2], H has polynomial growth. Hence, by [22, Theorem 3.11], G has polynomial growth. Thus, G^{fin} also has polynomial growth. Therefore, G is polynomially full by Lemma 5.2.4. □

5.3 Explicit Formulas of $\mathcal{F}_G$

In this subsection, we derive formulas for the number $\mathcal{F}_G$ for some polynomially full groups. Recall that when G is polynomially full, the groups $K_0^{fin}(C_r^*G)$ and $K_0^{fin}(C^*G)$ are free abelian with rank $\mathcal{F}_G$.

In the following, we give a formula for $\mathcal{F}_G$ for a finite abelian group G.

Proposition 5.3.1. *[12] For $G = \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k$, we have the following formula*

$$\mathcal{F}_G = \sum_{d_1|n_1} \cdots \sum_{d_k|n_k} \frac{\phi(d_1)\cdots\phi(d_k)}{\phi(\mathrm{lcm}(d_1,\cdots,d_k))},$$

where ϕ denotes Euler's totient function, lcm *denotes the least common multiple function, and the sums run over positive divisors d_i's of n_i's.*

Proof. Let's define an equivalence relation $\sim$ on $G = \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k$, which is coarser than $\sim_{\text{fin}}$:

We say $(x_1, \cdots, x_k) \sim (x_1', \cdots, x_k')$ if and only if, for all $i \in \{1, \cdots, k\}$ $x_i \sim_{\text{fin}} x_i'$ in $\mathbb{Z}/n_i$. It is easy to see that, $\sim$ is an equivalence relation on G.

Since homomorphic images of equivalent elements are equivalent, by looking at the projections to components, we can conclude that $(x_1, \cdots, x_k) \sim_{\text{fin}} (y_1, \cdots, y_k)$ implies $(x_1, \cdots, x_k) \sim (y_1, \cdots, y_k)$, for all $(x_1, \cdots, x_k), (y_1, \cdots, y_k) \in G$. So $\sim$ is coarser than $\sim_{\text{fin}}$.

For all $d_1, \cdots, d_k \in \mathbb{N}$ with $d_1|n_1, \cdots, d_k|n_k$, define

$$G_{d_1,\cdots,d_k} := \{(x_1, \cdots, x_k) \in G : \gcd(n_i, x_i) = \frac{n_i}{d_i} \text{ for } i \in \{1, \cdots, k\}\}.$$

It is easy to see that $G_{d_1,\cdots,d_k} = [(x_1,\cdots,x_k)]_\sim$ for all $(x_1,\cdots,x_k) \in G_{d_1,\cdots,d_k}$, where $[(x_1,\cdots,x_k)]_\sim$ is the equivalence class of the element $(x_1,\cdots,x_k)$ with respect to $\sim$.

Now, for all $(x_1,\cdots,x_k) \in G_{d_1,\cdots,d_k}$, we have

$$\begin{aligned} |[(x_1,\cdots,x_k)]_{\sim_{\text{fin}}}| &= \phi(\text{order}((x_1,\cdots,x_k))) \\ &= \phi(\text{lcm}(\text{order}(x_1),\cdots,\text{order}(x_k))) \\ &= \phi(\text{lcm}(d_1,\cdots,d_k)) \end{aligned}$$

and we have $|G_{d_1,\cdots,d_k}| = \phi(d_1)\cdots\phi(d_k)$. Hence, we have $|G_{d_1,\cdots,d_k}/_{\sim_{\text{fin}}}| = \frac{\phi(d_1)\cdots\phi(d_k)}{\phi(\text{lcm}(d_1,\cdots,d_k))}$. Thus, we get

$$\begin{aligned} \mathcal{F}_G &= |G^{\text{fin}}/_{\sim_{\text{fin}}}| \\ &= |G/_{\sim_{\text{fin}}}| \\ &= \sum_{d_1|n_1}\cdots\sum_{d_k|n_k} |G_{d_1,\cdots,d_k}/_{\sim_{\text{fin}}}| \\ &= \sum_{d_1|n_1}\cdots\sum_{d_k|n_k} \frac{\phi(d_1)\cdots\phi(d_k)}{\phi(\text{lcm}(d_1,\cdots,d_k))}. \end{aligned}$$

□

In the following, we give a formula for $\mathcal{F}_G$ for a finitely generated abelian group G.

Corollary 5.3.2. *[12] For $G = \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k \times \mathbb{Z}^m$, we have the following formula*

$$\mathcal{F}_G = \sum_{d_1|n_1}\cdots\sum_{d_k|n_k} \frac{\phi(d_1)\cdots\phi(d_k)}{\phi(\text{lcm}(d_1,\cdots,d_k))},$$

where the sums run over positive divisors d_i's of n_i's.

Proof. Let $H = \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k$. Since G is abelian and $G^{\text{fin}} \cong H$, result follows from Proposition 5.3.1. □

Remark 5.3.3. If we take $G = \mathbb{Z}/n$, then the above formula tells that $\mathcal{F}_G$ is equal to the number of positive divisors of n.

In the following, we give a formula for $\mathcal{F}_G$ for a dihedral group G.

Proposition 5.3.4. *[12] For $G = D_n$, we have*

$$\mathcal{F}_G = \begin{cases} \mathcal{F}_{\mathbb{Z}/n} + 1 & \textit{if } n \textit{ is odd} \\ \mathcal{F}_{\mathbb{Z}/n} + 2 & \textit{otherwise} , \end{cases}$$

where D_n is the dihedral group of order $2n$.

Proof. Let x, y be the generators of D_n with $x^2 = y^n = (xy)^2 = 1$. For all $a, b \in \mathbb{Z}$, we have $(xy^a) \cdot y^b \cdot (xy^a)^{-1} = (xy^a) \cdot y^b \cdot y^{-a}x = x^2y^{-b} = y^{-b}$, and $y^a \cdot y^b \cdot y^{-a} = y^b$. So for all $c \in \mathbb{Z}$, we get $[y^c]_{\sim_{\text{fin}}} \subseteq \{1, y, \cdots, y^{n-1}\}$, where $[y^c]_{\sim_{\text{fin}}}$ denotes the equivalence class of y^c in D_n.

Now let's show that $y^a \sim_{\text{fin}} y^b$ if and only if $\gcd(n, a) = \gcd(n, b)$:

For the forward direction, we have

$$\begin{aligned} y^a \sim_{\text{fin}} y^b &\implies \text{order}(y^a) = \text{order}(y^b) \\ &\implies \gcd(n, a) = \gcd(n, b). \end{aligned}$$

For the converse, assume we have $d = \gcd(n, a) = \gcd(n, b)$ for some $d \in \mathbb{N}$. So we get $\text{order}(y^a) = \text{order}(y^b) = \frac{n}{d}$ and $\gcd(\frac{n}{d}, \frac{a}{d}) = \gcd(\frac{n}{d}, \frac{b}{d}) = 1$. Hence, there exists $c \in \mathbb{N}$ such that $c \cdot \frac{a}{d} \equiv \frac{b}{d} \pmod{\frac{n}{d}}$. Thus, we get $ac \equiv b \pmod{n}$. So $(y^a)^c = y^{ac} = y^b$. Therefore, we get $y^a \sim_{\text{fin}} y^b$.

Now for all $a, b \in \mathbb{Z}$, xy^a has order 2, and

$$\begin{aligned}(xy^b) \cdot xy^a \cdot (xy^b)^{-1} &= (xy^b) \cdot xy^a \cdot y^{-b}x \\ &= xy^b xy^{a-b}x \\ &= xy^b x^2 y^{b-a} \\ &= xy^b y^{b-a} \\ &= xy^{2b-a}\end{aligned}$$

and

$$\begin{aligned}y^b \cdot xy^a \cdot y^{-b} &= xy^{-b}y^{a-b} \\ &= xy^{a-2b}.\end{aligned}$$

So we get

$$\begin{aligned}[xy^a]_{\sim_{\text{fin}}} &= \{xy^{a+2b} \mid b \in \mathbb{Z}\} \cup \{xy^{-a+2b} \mid b \in \mathbb{Z}\} \\ &= \{xy^{a+2b} \mid b \in \mathbb{Z}\}.\end{aligned}$$

Hence, $xy^a \sim_{\text{fin}} xy^b$ if and only if $\exists c \in \mathbb{Z}$ such that $a + 2c \equiv b \pmod{n}$. Thus, we get

$$[xy^a]_{\sim_{\text{fin}}} = \begin{cases} \{xy^b \mid b \in \mathbb{Z}\} & \text{if } n \text{ is odd} \\ \{xy^{a+2k} \mid k \in \mathbb{Z}\} & \text{otherwise} \end{cases}$$

Hence, we have

$$\mathcal{F}_G = \begin{cases} \mathcal{F}_{\mathbb{Z}/n} + 1 & \text{if } n \text{ is odd} \\ \mathcal{F}_{\mathbb{Z}/n} + 2 & \text{otherwise} \end{cases}$$

. $\square$

Remark 5.3.5. Let D_∞ be the infinite dihedral group. Let x, y be the elements generating D_∞ with relations $x^2 = (xy)^2 = 1$. Since D_∞ is virtually nilpotent (it contains the subgroup $\langle y \rangle \cong \mathbb{Z}$ of index 2), it is polynomially full by Corollary 5.2.5. Straightforward calculation shows that $\{1, x, xy\}$ is a complete set of representatives for the equivalence classes in $D_\infty^{\text{fin}}/_{\sim_{\text{fin}}}$. Hence, $\mathcal{F}_{D_\infty} = 3$.

In the following, we give a formula of $\mathcal{F}_G$ for $G = \mathfrak{S}_n$.

Proposition 5.3.6. *[12] For all $n \in \mathbb{N}$, $\mathcal{F}_{\mathfrak{S}_n}$ is equal to the number of conjugacy classes in $\mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on a finite set of n symbols.*

Proof. For all $g \in \mathfrak{S}_n$ and for all $a \in \mathbb{N}$ with $\gcd(a, \operatorname{order}(g)) = 1$, the permutations g and g^a have the same cycle structures. So they are conjugates. Now, $\forall g, h \in \mathfrak{S}_n$, we have $g \sim_{\text{fin}} h$ if and only if $\exists a \in \mathbb{N}$ with $\gcd(a, \operatorname{order}(g)) = 1$ and $g^a \in C^{\mathfrak{S}_n}(h)$. Therefore, we get $g \sim_{\text{fin}} h$ if and only if $g \in C^{\mathfrak{S}_n}(h)$. $\square$

6. CONCLUSIONS AND FUTURE DIRECTIONS

In summary, we use seminorms on $\mathbb{C}G$ to complete it to a smooth and dense sub-algebra of C_r^*G. We then lift the ordinary traces on $\mathbb{C}G$ to traces on the smooth dense sub-algebra to detect elements in the K_0 groups of C_r^*G and C^*G. Using those elements and combining with the results of Weinberger and Yu [2] we derive concrete lower bounds for the ranks of the groups $S(M)$ and $P(M)$. Then we introduce a class of groups called "polynomially full groups" for which the lower and the upper bounds in our main result are the same. We examine some closure properties of those groups and derive formulas for the numerical invariant $\mathcal{F}_G$ for some polynomially full G's.

In the following, we discuss some future directions to improve our results.

6.1 Different Growth Types

In our work, we lift the traces corresponding to the conjugacy classes with polynomial growth to traces on a suitable smooth and dense sub-algebra of C_r^*G. We can study different growth types such as sub-exponential and exponential growth. A result in the exponential growth case would include all conjugacy classes (in a finitely generated group all conjugacy classes has at most exponential growth). Hence, such a result would be spectacular. Sub-exponential growth case seems more plausible.

6.2 More Projections

In our work, we only use projections of the form $p_g = \frac{1+g+g^2+\cdots+g^{d-1}}{d} \in \mathbb{C}G \subseteq C^*G$, where g is an element in G with order d. We can get more projections in $\mathbb{C}G$ using the finite dimensional representations of the finite sub-groups of G. If we pair more projections with more traces we can improve the lower bound we derive for the ranks of the groups $S(M)$ and $P(M)$.

6.3 Higher Traces

In addition to the ordinary traces we have used, we can use higher traces (and lift them to suitable smooth and dense sub-algebras of C_r^*G) to detect more elements from the K_0 group. That result would improve the lower bounds we give to the groups $S(M)$ and $P(M)$.

6.4 Non-trivial Examples of Polynomially Full Groups

We know that the class of polynomially full groups is closed under the operations of

- taking finite cross products
- taking finitely generated subgroups
- taking finite extensions
- taking images with finite kernels

We also know that the class of polynomially full groups contains

- all virtually nilpotent groups
- all finitely generated torsion free groups

At this point, we don't know any example of a polynomially full group which cannot be obtained using those two classes and the operations above. Future work would include searching for such examples.

REFERENCES

[1] P. Baum, A. Connes, and N. Higson, Classifying space for proper actions and K-theory of group C^*-algebras, in *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, vol. 167 of *Contemp. Math.*, pp. 240–291, Amer. Math. Soc., Providence, RI, 1994.

[2] S. Weinberger and G. Yu, Finite part of operator K-theory for groups finitely embeddable into Hilbert space and the degree of nonrigidity of manifolds, *Geom. Topol.*, vol. 19, no. 5, pp. 2767–2799, 2015.

[3] A. Connes, *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994.

[4] A. Connes and H. Moscovici, Cyclic cohomology, the Novikov conjecture and hyperbolic groups, *Topology*, vol. 29, no. 3, pp. 345–388, 1990.

[5] G. G. Kasparov, Equivariant KK-theory and the Novikov conjecture, *Invent. Math.*, vol. 91, no. 1, pp. 147–201, 1988.

[6] N. Higson and G. Kasparov, E-theory and KK-theory for groups which act properly and isometrically on Hilbert space, *Invent. Math.*, vol. 144, no. 1, pp. 23–74, 2001.

[7] V. Lafforgue, Banach KK-theory and the Baum-Connes conjecture, in *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pp. 795–812, Higher Ed. Press, Beijing, 2002.

[8] I. Mineyev and G. Yu, The Baum-Connes conjecture for hyperbolic groups, *Invent. Math.*, vol. 149, no. 1, pp. 97–122, 2002.

[9] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, *Invent. Math.*, vol. 139, no. 1, pp. 201–240, 2000.

[10] G. J. Murphy, *C^*-algebras and operator theory*. Academic Press, Inc., Boston, MA, 1990.

[11] M. Rø rdam, F. Larsen, and N. Laustsen, *An introduction to K-theory for C^*-algebras*, vol. 49 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2000.

[12] S. K. Samurkaş, Bounds for the rank of the finite part of operator K-Theory, *to appear in Journal of Noncommutative Geometry*.

[13] J. Roe, An index theorem on open manifolds. I, II, *J. Differential Geom.*, vol. 27, no. 1, pp. 87–113, 115–136, 1988.

[14] A. Engel, Banach strong Novikov conjecture for polynomially contractible groups, *to appear in Advances in Mathematics*.

[15] L. B. Schweitzer, A short proof that $M_n(A)$ is local if A is local and Fréchet, *Internat. J. Math.*, vol. 3, no. 4, pp. 581–589, 1992.

[16] J. M. Gracia-Bondí a, J. C. Várilly, and H. Figueroa, *Elements of noncommutative geometry*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Boston, Inc., Boston, MA, 2001.

[17] J. Stallings, Centerless groups—an algebraic formulation of Gottlieb's theorem, *Topology*, vol. 4, pp. 129–134, 1965.

[18] S. Gong, Finite part of operator K-theory for groups with rapid decay, *J. Noncommut. Geom.*, vol. 9, no. 3, pp. 697–706, 2015.

[19] G. Yu, The Novikov conjecture for algebraic K-theory of the group algebra over the ring of Schatten class operators, *Adv. Math.*, vol. 307, pp. 727–753, 2017.

[20] A. Ranicki, *Algebraic and geometric surgery*. Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2002. Oxford Science Publications.

[21] S. Weinberger, Z. Xie, and G. Yu, Additivity of higher rho invariants and nonrigidity of topological manifolds, *ArXiv e-prints*, Aug. 2016.

[22] J. A. Wolf, Growth of finitely generated solvable groups and curvature of Riemanniann manifolds, *J. Differential Geometry*, vol. 2, pp. 421–446, 1968.

[23] M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Études Sci. Publ. Math.*, no. 53, pp. 53–73, 1981.